IMSC 2048 Practice Midterm

February 27, 2026

Excercise 1. *Let V be a finite-dimensional vector space over a field $\mathbb{R}$. Let $B : V \times V \to \mathbb{R}$ be a non-degenerate bilinear form. Suppose that for all $x, y \in V$, $B(x, y) = 0$ implies $B(y, x) = 0$. Prove that B is either symmetric or skew-symmetric.*

Excercise 2. *Define the center of character χ as $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$. Prove that $Z(\chi)$ is a normal subgroup of G. If χ is irreducible, show that $Z(\chi)$ contains the center of G.*

Excercise 3. *Let $G = S_4$ be the symmetric group on $\{1, 2, 3, 4\}$. Recall the character table of S_4: the corresponding conjugacy classes are represented by the cycle types* (1), (12), (12)(34), (123), *and* (1234), *and the numbers in the second row are the sizes of the conjugacy classes. The corresponding representations are denoted by V_1, V_2, V_3, V_4, V_5.*

S_4	(1)	(12)	(12)(34)	(123)	(1234)
	1	6	3	8	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

(a) Decompose the tensor product $V_4 \otimes V_4$ into irreducible representations.

(b) Determine all normal subgroups of S_4 from the character table.

Excercise 4. *Let $A \in M_{m\times n}(\mathbb{R})$ with singular value decomposition $A = QDP^T$, where $Q \in \mathrm{O}(m)$, $P \in \mathrm{O}(n)$, and D has diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.*

(a) Compute the singular values of A.

(b) Compute the singular value decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) *Find the rank-*1 *matrix* B *that minimizes* $\|A - B\|_F$*, and compute* $\|A - B\|_F$*. Here* $\|M\|_F = \sqrt{\sum_{i,j} m_{ij}^2}$ *for* $M = (m_{ij})$ *is the Frobenius norm.*

Excercise 5. *Consider the vector space of* $n \times n$ *real matrices, equipped with the symmetric bilinear form* $B(X, Y) = \mathrm{tr}(XY)$*. Determine the signature of* B*.*